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Anti-synchronization of fractional order chaotic and hyperchaotic systems with fully unknown parameters using modified adaptive control

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Abstract: The objective of this article is to implement and extend applications of adaptive control to anti-synchronize different fractional order chaotic and hyperchaotic dynamical systems. The sufficient conditions for achieving anti–synchronization are derived by using the Lyapunov stability theory and an analytic expression of the controller with its adaptive laws of parameters is shown. Theoretical analysis and numerical simulations are shown to verify the results.

Keywords: anti-synchronization, adaptive control, fractional order hyperchaos, fractional order chaos, unknown parameters.

PACS: 05.45.-a, 47.52.+j, 89.75.-k

1 Introduction

We have seen some dramatic and amazing developments in the research of fractional order chaotic dynamics, particularly with respect to their interaction with the other fields of research and applications. There is now a developed science of fractional order chaos that has a vitally strong interaction between theory and experiment. This is a great leap compared to previously, in which theoretical work existed largely in the absence of substantial experimental realizations. This subject has gained much interest and appreciation in practical applications such as viscoelastic systems, electrode-electrolyte polarization, electromagnetic waves, etc. [1, 2]. Issues such as synchronization of fractional order chaotic systems in a broad variety of situations and the use of fractional order chaotic dynamics for various purposes are at the forefront of recent applications in nonlinear science. This topic encompasses a common link, which is uniting the knowledge of basic mathematical properties of fractional order chaos and specific practical considerations of various applications [3, 4]. A wide variety of approaches have been proposed for the synchronization of fractional order chaotic systems such as adaptive control, sliding mode control, linear active control technique, projective synchronization, and nonlinear active control [5–30]. Another interesting phenomenon discovered is anti–synchronization (AS) which is noticeable in periodic oscillators. In the AS phenomenon, the state vectors of the synchronized systems have the same amplitude but opposite signs as those of the driving system. Thus, the sum of two signals are expected to converge to zero when AS appears. Several control methods have been applied to anti–synchronize chaotic systems [31–35]. Fortunately, some existing methods of anti–synchronizing of integer order can be generalized to anti-synchronize fractional order chaotic systems through some rigorous mathematical theory. However, in practical engineering situations, the parameters are probably unknown and may change from time to time. Therefore there is a vital need to effectively anti-synchronize two chaotic systems (identical and different) with unknown parameters. This is typically important in theoretical research as well as in practical applications. Among the aforementioned methods, the adaptive control strategy is an efficient control method to anti–synchronize fractional order chaotic chaotic systems. The adaptive control method is used when some or all parameters of the chaotic systems are unknown. The significant features of the adaptive control strategy include fast response, robustness against perturbations, good transient performance, and easy implementation in real applications. In (2013), Agrawal et. al [14] developed a novel adaptive synchronization scheme associated with the parameter update rule for identical and nonidentical fractional order chaotic systems with unknown parameters for the synchronization of fractional order as well as inte-

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ger order chaotic systems. In spite of [14, 33–35], the epitome of this paper centers on fractional order chaos anti–synchronization between two fractional order chaotic systems, also the aforementioned method concerns synchronizing two fractional order chaotic system with low dimensional attractors characterized by one positive Lyapunov exponent. This feature limits the complexity of the fractional order chaotic dynamics. It is believed that fractional order chaotic systems with higher dimensional attractors have much wider applications. The rest of the paper is organized as follows. In section 2, we briefly describe the problem. In sections 3, we present the adaptive anti-synchronization scheme with a parameter update law for two different chaotic systems. Section 4, presents the adaptive anti-synchronization scheme with a parameter update law for two different hyperchaotic systems. The conclusion is given at the end.

2 Problem formulation

2.1 Preliminaries of fractional-order calculus

There are several definitions of fractional derivatives, the commonly used definition is the Riemann-Liouville definition, as follows:

$$ {}_aD_t^q z(t) = \frac{d^n}{dt^n} J_t^{n-q} z(t), \quad q > 0, \tag{1}$$

where $n = \lceil q \rceil$, and

$$J_t^\vartheta \psi(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t \frac{\psi(\upsilon)}{(t-\upsilon)^{1-\vartheta}}\, d\upsilon, \tag{2}$$

where $0 < \vartheta \leq 1$ and $\Gamma(.)$ is gamma function. The Caputo differential operator of fractional order q is defined as

$$ {}^cD_t^q z(t) = J_t^{n-q} z^n(t), \quad q > 0, \tag{3}$$

where $n = \lceil q \rceil$.

Lemma 1. [1, 14] *In Riemann–Liouville derivatives if $p > q \geq 0$, m and n are integers such that $0 \leq m-1 \leq p < m$, $0 \leq n-1 < n$, then we obtain*

$$ {}_aD_t^q \left({}_aD_t^{-q} f(t) \right) = {}_aD_t^{p-q} f(t). \tag{4}$$

Lemma 2. [1, 14] *In Riemann–Liouville derivatives if $p > q \geq 0$, m and n are integers such that $0 \leq m-1 \leq p < m$, $0 \leq n-1 \leq q < n$, then we obtain*

$$ {}_aD_t^p \left({}_aD_t^q f(t) \right) = {}_aD_t^{p+q} f(t) \tag{5}$$

$$ - \sum_{J=1}^{n} \left[{}_aD_t^{q-J} f(t) \right]_{t=a} \times \frac{(t-a)^{-p-J}}{\Gamma(1-p-J)}.$$

2.2 Modified adaptive anti–synchronization

Consider the chaotic master system described by

$$D_t^q x = f(x) + F(x)\alpha, \tag{6}$$

where $x \in \Omega_1 \subset R^n$ is the state vector of system (6), $\alpha \in R^m$ is the unknown parameter vector of the system, $f(x)$ is an $n \times 1$ matrix, and $F(x)$ is an $n \times m$ matrix. Similarly, the slave system described by

$$D_t^q y = g(y) + G(y)\beta + u, \tag{7}$$

where $y \in \Omega_2 \subset R^n$ is the state vector of system (7), $\beta \in R^q$ is the unknown parameter vector of the system, $g(y)$ is an $n \times 1$ matrix, $G(y)$ is an $n \times q$ matrix, and $u \in R^n$ is a control input vector. $e = y + x$ is the anti–synchronization error vector. Our goal is to design a controller u such that the trajectory of the slave system (7) with initial conditions y_0 can asymptotically approach the master system (6) with initial conditions x_0 and finally implement anti–synchronization in the sense that $\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y(t) + x(t)\| = 0$, where $\|\cdot\|$ is the Euclidean norm.

Theorem 1. *If the nonlinear control function is selected as*

$$\begin{aligned} U = & -f(x) - F(x)\alpha - g(y) - G(y)\beta \\ & + D_t^{q-1}[F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) \\ & - \left(D_t^{q-1} e(t) \right) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))} - ke] \end{aligned} \tag{8}$$

and the adaptive laws of the parameters are taken as

$$\begin{aligned} \dot{\hat{\alpha}} &= [F(x)]^T e, \\ \dot{\hat{\beta}} &= [G(y)]^T e, \end{aligned} \tag{9}$$

where $\hat{\alpha} = \alpha - \tilde{\alpha}$, $\hat{\beta} = \beta - \tilde{\beta}$, $k > 0$ is a constant, $q \in [0, 1]$ is the order of the derivative, and $\tilde{\alpha}$, $\tilde{\beta}$ are the estimated parameters of α and β, respectively.

Proof. From Eqs. (7) and (6) we get the error dynamical system as follows:

$$D_t^q e(t) = g(y) + G(y)\beta + f(x) + F(x)\alpha + U. \tag{10}$$

Inserting (8) into (10) yields the following:

$$D_t^q e(t) = D_t^{q-1}[F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) \tag{11}$$

$$- (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))} - ke].$$

If a Lyapunov function candidate is chosen as

$$V(e, \hat{\alpha}, \hat{\beta}) = \frac{1}{2}[e^T e + (\alpha - \tilde{\alpha})^T(\alpha - \tilde{\alpha}) + (\beta - \tilde{\beta})^T(\beta - \tilde{\beta}))] \tag{12}$$

the time derivative of $V(e, \hat{\alpha}, \hat{\beta})$ along the trajectory of the error dynamical system (11) is

$$\dot{V}(e, \hat{\alpha}, \hat{\beta}) = [\dot{e}^T e + (\alpha - \tilde{\alpha})^T \dot{\tilde{\alpha}} + (\beta - \tilde{\beta})^T \dot{\tilde{\beta}}] \tag{13}$$

Using Lemma 2 in Eq. (13) we get

$$\begin{aligned}\dot{V}(e, \hat{\alpha}, \hat{\beta}) &= ([D_t^{q-1}(D_t^q e(t)) \\ &+ (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}] \\ &+ (\alpha - \tilde{\alpha})^T \dot{\tilde{\alpha}} + (\beta - \tilde{\beta})^T \dot{\tilde{\beta}})).\end{aligned} \tag{14}$$

From Eqs. (9) and (13), we get

$$\begin{aligned}\dot{V}(e, \hat{\alpha}, \hat{\beta}) &= [D_t^{q-1}(D_t^{q-1}[F(x)(\tilde{\alpha} - \alpha) \\ &+ G(y)(\tilde{\beta} - \beta) \\ &- (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))} - ke] \\ &+ (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}]^T \\ &+ (\alpha - \tilde{\alpha})^T \dot{\tilde{\alpha}} + (\beta - \tilde{\beta})^T \dot{\tilde{\beta}},\end{aligned} \tag{15}$$

since $\forall q \in [0, 1], (1 - q) > 0$ and $(q - 1) < 0$. Now, using Lemma 1 and Eq. (9), Eq. (15) reduces to

$$\begin{aligned}\dot{V}(e, \hat{\alpha}, \hat{\beta}) &= [(F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) \\ &- (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))} - ek) \\ &+ (D_t^{q-1} e(t)) \frac{(t)^{-(q-1)-1}}{\Gamma(-(q-1))}]^T e \\ &- (\alpha - \tilde{\alpha})^T([F(x)]^T e) \\ &- (\beta - \tilde{\beta})^T([G(y)]^T e) \\ &= [(\alpha - \tilde{\alpha})^T F(x)^T \\ &+ (\beta - \tilde{\beta})^T G(y)^T - ke^T]e \\ &- (\alpha - \tilde{\alpha})^T [F(x)]^T e \\ &- (\beta - \tilde{\beta})^T [G(y)]^T e \\ &= -ke^T e \leq 0.\end{aligned} \tag{16}$$

Since V and $\dot{V}$ are positive and negative semi-definite respectively, therefore, according to the Lyapunov stability theory [36], the response system (7) is both globally and asymptotically anti–synchronized to the drive system (6). This completes the proof. □

3 Adaptive anti–synchronization between two fractional-order chaotic systems

In order to achieve the behavior of anti–synchronization between two fractional-order chaotic systems using modified adaptive control, we take the fractional-order chaotic Lü [35] system to be the drive system and the fractional-order chaotic Liu system [22] to be the response system. The variables of the drive system are represented by the subscript 1 and the response system by the subscript 2. Both systems are described respectively by the following equations:

$$\begin{aligned}\frac{d^\alpha x_1}{d^\alpha t} &= a_1(y_1 - x_1), \\ \frac{d^\alpha y_1}{d^\alpha t} &= -x_1 z_1 + c_1 y_1, \\ \frac{d^\alpha z_1}{d^\alpha t} &= x_1 y_1 - b_1 z_1,\end{aligned} \tag{17}$$

and

$$\begin{aligned}\frac{d^\alpha x_2}{d^\alpha t} &= a_2(y_2 - x_2) + u_1, \\ \frac{d^\alpha y_2}{d^\alpha t} &= b_2 x_2 - x_2 z_2 + u_2, \\ \frac{d^\alpha z_2}{d^\alpha t} &= -c_2 z_2 + d_2 x_2^2 + u_3,\end{aligned} \tag{18}$$

where, $U = (u_1, u_2, u_3)^T$ is the control function to be designed. In order to determine the control functions to realize adaptive anti–synchronization between the systems in Eqs. (17) and (18), we add (17) to (18) and obtain

$$\begin{aligned}D_t^{q_1} e_1(t) &= a_2(y_2 - x_2) + a_1(y_1 - x_1) + u_1, \\ D_t^{q_2} e_2(t) &= b_2 x_2 - x_2 z_2 - x_1 z_1 + c_1 y_1 + u_2, \\ D_t^{q_3} e_3(t) &= -c_2 z_2 + d_2 x_2^2 + x_1 y_1 - b_1 z_1 + u_3,\end{aligned} \tag{19}$$

where $e_1 = x_2 + x_1$, $e_2 = y_2 + y_1$, and $e_3 = z_2 + z_1$. Our goal is to derive the controller U with a parameter estimation update law such that Eqs. (18) globally and asymptotically anti-synchronize Eqs.(17).

Theorem 2. *The fractional-order chaotic Liu system (18) can anti–synchronize the fractional-order Lü system (17) globally and asymptotically for any different initial condition with the following adaptive controller:*

$$\begin{aligned}u_1 &= -a_2(y_2 - x_2) - a_1(y_1 - x_1) \\ &+ D_t^{q_1-1}[\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) \\ &- (D_t^{q_1-1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1],\end{aligned} \tag{20}$$

$$u_2 = -b_2x_2 + x_2z_2 + x_1z_1 - c_1y_1 + D_t^{q_2-1}[\tilde{b}_2x_2 + \tilde{c}_1y_1 - \left(D_t^{q_2-1}e_2(t)\right) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2\Bigg],$$

$$u_3 = c_2z_2 - d_2x_2^2 - x_1y_1 + b_1z_1 + D_t^{q_3-1}[-\tilde{c}_2z_2 + \tilde{d}_2x_2^2 - \tilde{b}_1z_1 - (D_t^{q_3-1}e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3],$$

and parameter update rules

$$\dot{\hat{a}}_1 = (y_1 - x_1)e_1, \quad (21)$$
$$\dot{\hat{b}}_1 = -z_1e_3,$$
$$\dot{\hat{c}}_1 = y_1e_2,$$
$$\dot{\hat{a}}_2 = (y_2 - x_2)e_1,$$
$$\dot{\hat{b}}_2 = x_2e_2,$$
$$\dot{\hat{c}}_2 = -z_2e_3,$$
$$\dot{\hat{d}}_2 = x_2^2e_3,$$

where $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{a}_2, \hat{b}_2, \hat{c}_2$, and $\hat{d}_2$ are estimates of $a_1, b_1, c_1, a_2, b_2, c_2$, and d_2, respectively.

Proof. Applying the control law equation (20) to Eq. (19) yields the resulting closed-loop error dynamical system as follows:

$$D_t^{q_1}e_1(t) = D_t^{q_1-1}[\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) - (D_t^{q_1-1}e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1], \quad (22)$$

$$D_t^{q_2}e_2(t) = D_t^{q_2-1}[\tilde{b}_2x_2 + \tilde{c}_1y_1 - (D_t^{q_2-1}e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2],$$

$$D_t^{q_3}e_3(t) = D_t^{q_3-1}[-\tilde{c}_2z_2 + \tilde{d}_2x_2^2 - \tilde{b}_1z_1 - (D_t^{q_3-1}e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3],$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$, $\tilde{b}_1 = b_1 - \hat{b}_1$, $\tilde{c}_1 = c_1 - \hat{c}_1$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{b}_2 = b_2 - \hat{b}_2$, $\tilde{c}_2 = c_2 - \hat{c}_2$, and $\tilde{d}_2 = d_2 - \hat{d}_2$.

Consider the following Lyapunov function candidate:

$$V = \frac{1}{2}(e^Te + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2 + \tilde{c}_2^2 + \tilde{d}_2^2), \quad (23)$$

then the time derivative of V along the solution of the error dynamical system equation (22) gives

$$\dot{V} = (e^Te + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{d}_2\dot{\tilde{d}}_2). \quad (24)$$

Using Lemma 2 in Eq. (24) we get

$$\begin{aligned}\dot{V} &= ([D_t^{1-q_1}(D_t^{q_1}e_1(t)) + (D_t^{q_1}e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))}]e_1 \\ &+ ([D_t^{1-q_2}(D_t^{q_2}e_2(t)) + (D_t^{q_2}e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))}]e_2 \\ &+ ([D_t^{1-q_3}(D_t^{q_3}e_3(t)) + (D_t^{q_3}e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))}]e_3 \\ &+ \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{d}_2\dot{\tilde{d}}_2 \\ &= ([D_t^{1-q_1}(D_t^{q_1-1}[\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) - (D_t^{q_1-1}e_1(t)) \\ &\times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1]) + (D_t^{q_1}e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))}]e_1 \\ &+ ([D_t^{1-q_2}(D_t^{q_2-1}[\tilde{b}_2x_2 + \tilde{c}_1y_1 - (D_t^{q_2-1}e_2(t)) \\ &\times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2]) + (D_t^{q_2}e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))}]e_2 \\ &+ ([D_t^{1-q_3}(D_t^{q_3-1}[-\tilde{c}_2z_2 + \tilde{d}_2x_2^2 - \tilde{b}_1z_1 - (D_t^{q_3-1}e_3(t)) \\ &\times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3]) \\ &+ (D_t^{q_3}e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))}]e_3 + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 + \tilde{c}_1\dot{\tilde{c}}_1 \\ &+ \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{d}_2\dot{\tilde{d}}_2,\end{aligned} \quad (25)$$

since $\forall q \in [0, 1]$, $(1 - q) > 0$ and $(q - 1) < 0$. Now, using Lemma 1, Eq. (25) reduces to

$$\begin{aligned}\dot{V} &= [\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) - e_1]e_1 \\ &+ [\tilde{b}_2x_2 + \tilde{c}_1y_1 - e_2]e_2 \\ &+ [-\tilde{c}_2z_2 + \tilde{d}_2x_2^2 - \tilde{b}_1z_1 - e_3]e_3 \\ &+ \tilde{a}_1(-(y_1 - x_1)e_1) + \tilde{b}_1(z_1e_3) + \tilde{c}_1(-y_1e_2) \\ &+ \tilde{a}_2(-(y_2 - x_2)e_1) + \tilde{b}_2(-x_2e_2) + \tilde{c}_2(z_2e_3) \\ &+ \tilde{d}_2(-x_2^2e_3), \\ &= -e^Te \le 0.\end{aligned} \quad (26)$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of the zero solution of the system equation (22), it follows that $\lim_{t\to\infty} \|e(t)\| = 0$. Therefore system (18) can anti–synchronize system (17) asymptotically. This completes the proof. □

4 Numerical simulations

In the numerical simulations, the Adams-Bashforth-Moulton method is used to solve the systems. The frac-

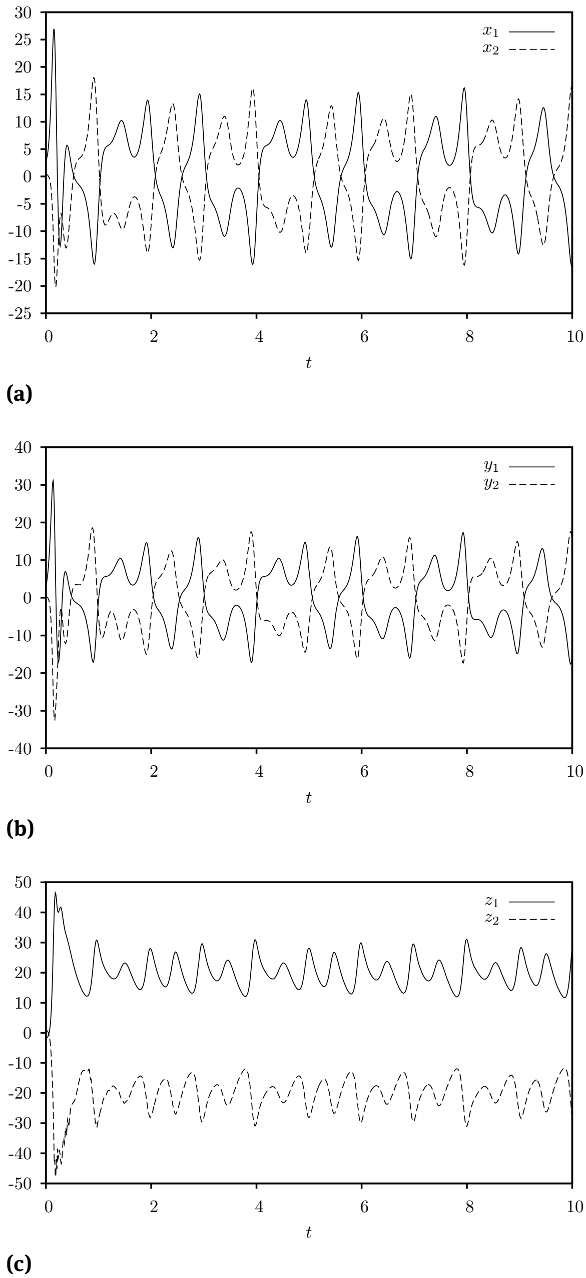


Figure 1: State trajectories of drive system (17) and response system (18): **(a)**: Signals x_1 and x_2; **(b)**: signals y_1 and y_2; and **(c)**: signals z_1 and z_2.

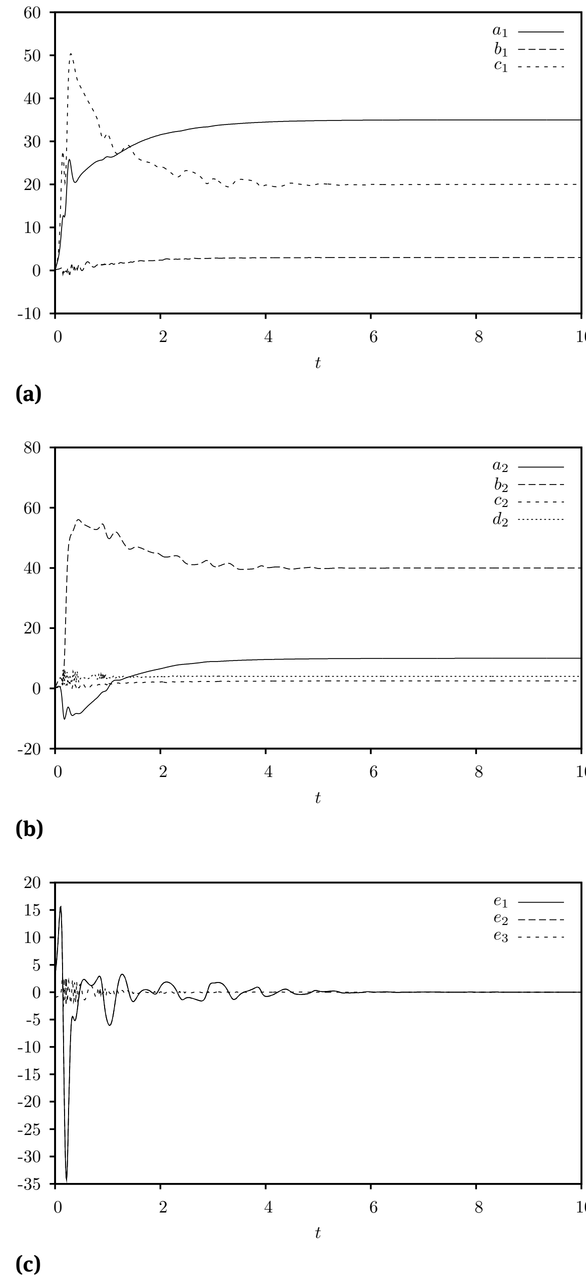


Figure 2: (a)–(b): Change in parameters a_1, b_1, c_1 and a_2, b_2, c_2, d_2 of the drive system (17) and the response system (18) with time t. **(c)**: The error signals $e_1; e_2; e_3$ of the drive system (17) and the response system (18) under the controller (20) and the parameters update law (21).

tional order is chosen as $\alpha = 0.95$, and the unknown parameters are chosen as $a_1 = 35$, $b_1 = 3$, $c_1 = 20$, and $a_2 = 10$, $b_2 = 40$, $c_2 = 2.5$, and $d_2 = 4$, so that both systems exhibit a chaotic behavior. The initial values of the fractional-order drive system (17), the fractional-order response system (18) and the estimated parameters are arbitrarily chosen in the simulations as $(x_1(0) = 0.2, y_1(0) = 0.6, z_1(0) = 1)$, $(x_2(0) = 7, y_2(0) = 11, z_2(0) = 15)$, and $\hat{a}_1(0) = 0.2$, $\hat{b}_1(0) = 0.2$, $\hat{c}_1(0) = 0.2$ and $\hat{a}_2(0) = 0.2$, $\hat{b}_2(0) = 0.2$, $\hat{c}_2(0) = 0.2$, $\hat{d}_2(0) = 0.2$, respectively. Anti–synchronization of the systems (17) and (18) via the adaptive control law (20) and (21) are shown in Figs. (1)–(2). Fig. (1) displays the state trajectories of the drive system (17) and response system (18). Fig. (2) (a)–(b) show that the estimates $\hat{a}_1(t)$, $\hat{b}_1(t)$, $\hat{c}_1(t)$ and $\hat{a}_2(t)$, $\hat{b}_2(t)$, $\hat{c}_2(t)$, $\hat{d}_2(t)$ of the unknown parameters converge to $a_1 = 35$, $b_1 = 3$, $c_1 =$

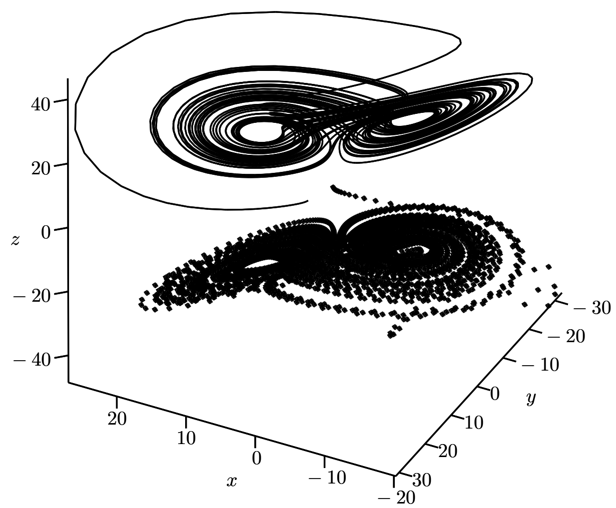


Figure 3: Fractional-order chaotic Lü system (solid line) and the controlled fractional-order Liu system (dotted line).

20 and $a_2 = 10, b_2 = 40, c_2 = 2.5, d_2 = 4$ as $t \to \infty$. Fig. (2) (c) displays the anti–synchronization errors of systems (17) and (18). Fig. (3) shows that the fractional order Liu system is controlled to be the fractional order Lü system.

5 Adaptive anti–synchronization between two fractional-order hyperchaotic systems

In order to achieve the behavior of anti–synchronization between two fractional-order chaotic systems using modified adaptive control, we take the fractional-order hyperchaotic Lorenz system [37] to be the drive system and the fractional-order hyperchaotic Chen system [38] to be the response system. The variables of the drive system are represented by the subscript 1 and the response system by the subscript 2. Both of the systems are described respectively by the following equations:

$$\begin{aligned} \frac{d^\alpha x_1}{d^\alpha t} &= a_1(y_1 - x_1) + w_1, \\ \frac{d^\alpha y_1}{d^\alpha t} &= c_1 x_1 - x_1 z_1 - y_1, \\ \frac{d^\alpha z_1}{d^\alpha t} &= x_1 y_1 - b_1 z_1, \\ \frac{d^\alpha w_1}{d^\alpha t} &= -y_1 z_1 + r_1 w_1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \frac{d^\alpha x_2}{d^\alpha t} &= a_2(y_2 - x_2) + w_2 + u_1, \\ \frac{d^\alpha y_2}{d^\alpha t} &= r_2 x_2 - x_2 z_2 + c_2 y_2 + u_2, \\ \frac{d^\alpha z_2}{d^\alpha t} &= x_2 y_2 - b_2 z_2 + u_3, \\ \frac{d^\alpha w_2}{d^\alpha t} &= x_2 z_2 + d_2 w_2 + u_4, \end{aligned} \tag{28}$$

where $U = (u_1, u_2, u_3, u_4)^T$ is the control function to be designed. In order to determine the control functions to realize adaptive anti–synchronization between the systems in Eqs. (27) and (28) , we add (27) to (28) and obtain

$$\begin{aligned} D_t^{q_1} e_1(t) &= a_2(y_2 - x_2) + w_2 + a_1(y_1 - x_1) \\ &\quad + w_1 + u_1, \\ D_t^{q_2} e_2(t) &= r_2 x_2 - x_2 z_2 + c_2 y_2 + c_1 x_1 - x_1 z_1 \\ &\quad - y_1 + u_2, \\ D_t^{q_3} e_3(t) &= x_2 y_2 - b_2 z_2 + x_1 y_1 - b_1 z_1 + u_3, \\ D_t^{q_4} e_4(t) &= x_2 z_2 + d_2 w_2 - y_1 z_1 + r_1 w_1 + u_4, \end{aligned} \tag{29}$$

where $e_1 = x_2 + x_1, e_2 = y_2 + y_1$ $e_3 = z_2 + z_1$, and $e_4 = w_2 + w_1$. Our goal is to derive the controller U with a parameter estimation update law such that Eqs. (28) globally and asymptotically anti-synchronize Eqs.(27).

Theorem 3. *The fractional-order hyperchaotic Lorenz system (28) can anti–synchronize with the fractional-order Chen system (27) globally and asymptotically for any different initial condition with the following adaptive controller:*

$$\begin{aligned} u_1 &= -a_2(y_2 - x_2) - w_2 - a_1(y_1 - x_1) \\ &\quad - w_1 + D_t^{q_1 - 1}[\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) \\ &\quad - (D_t^{q_1 - 1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1 - 1))} - e_1], \\ u_2 &= -r_2 x_2 + x_2 z_2 - c_2 y_2 - c_1 x_1 + x_1 z_1 + y_1 \\ &\quad + D_t^{q_2 - 1}[\tilde{r}_2 x_2 + \tilde{c}_2 y_2 + \tilde{c}_1 x_1 - (D_t^{q_2 - 1} e_2(t)) \\ &\quad \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2 - 1))} - e_2], \\ u_3 &= -x_2 y_2 + b_2 z_2 - x_1 y_1 + b_1 z_1 \\ &\quad + D_t^{q_3 - 1}[-\tilde{b}_2 z_2 - \tilde{b}_1 z_1 - (D_t^{q_3 - 1} e_3(t)) \\ &\quad \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3 - 1))} - e_3], \\ u_4 &= -x_2 z_2 - d_2 w_2 + y_1 z_1 - r_1 w_1 \\ &\quad + D_t^{q_4 - 1}[\tilde{d}_2 w_2 + \tilde{r}_1 w_1 - (D_t^{q_4 - 1} e_4(t)) \\ &\quad \times \frac{(t)^{-(q_4-1)-1}}{\Gamma(-(q_4 - 1))} - e_4], \end{aligned} \tag{30}$$

and parameter update rules

$$\begin{aligned} \dot{\hat{a}}_1 &= (y_1 - x_1) e_1, \\ \dot{\hat{b}}_1 &= -z_1 e_3, \end{aligned} \tag{31}$$

$$\dot{\hat{c}}_1 = x_1 e_2,$$
$$\dot{\hat{r}}_1 = w_1 e_4,$$
$$\dot{\hat{a}}_2 = (y_2 - x_2)e_1,$$
$$\dot{\hat{b}}_2 = -z_2 e_3,$$
$$\dot{\hat{c}}_2 = y_2 e_2,$$
$$\dot{\hat{r}}_2 = x_2 e_2,$$
$$\dot{\hat{d}}_2 = w_2 e_4,$$

where $\hat{a}_1$, $\hat{b}_1$, $\hat{c}_1$, $\hat{r}_1$, $\hat{a}_2$, $\hat{b}_2$, $\hat{c}_2$, $\hat{r}_2$, and $\hat{d}_2$ are estimates of a_1, b_1, c_1,r_1, a_2, b_2, c_2, r_2, and d_2 respectively.

Proof. Applying the control law equation (30) to Eq. (29) yields the resulting closed-loop error dynamical system as follows:

$$\begin{aligned} D_t^{q_1} e_1(t) &= D_t^{q_1-1}[\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) \\ &\quad - (D_t^{q_1-1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1], \\ D_t^{q_2} e_2(t) &= D_t^{q_2-1}[\tilde{r}_2 x_2 + \tilde{c}_2 y_2 + \tilde{c}_1 x_1 \\ &\quad - (D_t^{q_2-1} e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2], \\ D_t^{q_3} e_3(t) &= D_t^{q_3-1}[-\tilde{b}_2 z_2 - \tilde{b}_1 z_1 \\ &\quad - (D_t^{q_3-1} e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3], \\ D_t^{q_4} e_4(t) &= D_t^{q_4-1}[\tilde{d}_2 w_2 + \tilde{r}_1 w_1 \\ &\quad - (D_t^{q_4-1} e_4(t)) \times \frac{(t)^{-(q_4-1)-1}}{\Gamma(-(q_4-1))} - e_4], \end{aligned} \tag{32}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$, $\tilde{b}_1 = b_1 - \hat{b}_1$, $\tilde{c}_1 = c_1 - \hat{c}_1$, $\tilde{r}_1 = r_1 - \hat{r}_1$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{b}_2 = b_2 - \hat{b}_2$, $\tilde{c}_2 = c_2 - \hat{c}_2$, $\tilde{r}_2 = r_2 - \hat{r}_2$, and $\tilde{d}_2 = d_2 - \hat{d}_2$

Consider the following Lyapunov function candidate:

$$\begin{aligned} V &= \frac{1}{2}(e^T e + \tilde{a}_1^2 + \tilde{b}_1^2 + \tilde{c}_1^2 + \tilde{r}_1^2 + \tilde{a}_2^2 \\ &\quad + \tilde{b}_2^2 + \tilde{c}_2^2 + \tilde{r}_2^2 + \tilde{d}_2^2), \end{aligned} \tag{33}$$

then the time derivative of V along the solution of the error dynamical system equation (32) gives

$$\begin{aligned} \dot{V} &= (e^T e + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{c}_1 \dot{\tilde{c}}_1 + \tilde{r}_1 \dot{\tilde{r}}_1 \\ &\quad + \tilde{a}_2 \dot{\tilde{a}}_2 + \tilde{b}_2 \dot{\tilde{b}}_2 + \tilde{c}_2 \dot{\tilde{c}}_2 + \tilde{r}_2 \dot{\tilde{r}}_2 + \tilde{d}_2 \dot{\tilde{d}}_2). \end{aligned} \tag{34}$$

Using Lemma 2 in Eq. (34) we get

$$\begin{aligned} \dot{V} &= ([D_t^{1-q_1}(D_t^{q_1} e_1(t)) + (D_t^{q_1} e_1(t)) \\ &\quad \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))}]e_1 \\ &\quad + ([D_t^{1-q_2}(D_t^{q_2} e_2(t)) + (D_t^{q_2} e_2(t)) \\ &\quad \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))}]e_2 \\ &\quad + ([D_t^{1-q_3}(D_t^{q_3} e_3(t)) + (D_t^{q_3} e_3(t)) \\ &\quad \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))}]e_3 \\ &\quad + ([D_t^{1-q_4}(D_t^{q_4} e_4(t)) + (D_t^{q_4} e_4(t)) \\ &\quad \times \frac{(t)^{-(q_4-1)-1}}{\Gamma(-(q_4-1))}]e_4 + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{c}_1 \dot{\tilde{c}}_1 + \tilde{r}_1 \dot{\tilde{r}}_1 \\ &\quad + \tilde{a}_2 \dot{\tilde{a}}_2 + \tilde{b}_2 \dot{\tilde{b}}_2 + \tilde{c}_2 \dot{\tilde{c}}_2 + \tilde{r}_2 \dot{\tilde{r}}_2 + \tilde{d}_2 \dot{\tilde{d}}_2 \\ &= ([D_t^{1-q_1}(D_t^{q_1-1}[\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) \\ &\quad - (D_t^{q_1-1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))} - e_1]) \\ &\quad + (D_t^{q_1} e_1(t)) \times \frac{(t)^{-(q_1-1)-1}}{\Gamma(-(q_1-1))}]e_1 \\ &\quad + ([D_t^{1-q_2}(D_t^{q_2-1}[\tilde{r}_2 x_2 + \tilde{c}_2 y_2 \\ &\quad + \tilde{c}_1 x_1 - (D_t^{q_2-1} e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))} - e_2]) \\ &\quad + (D_t^{q_2} e_2(t)) \times \frac{(t)^{-(q_2-1)-1}}{\Gamma(-(q_2-1))}]e_2 \\ &\quad + ([D_t^{1-q_3}(D_t^{q_3-1}[-\tilde{b}_2 z_2 - \tilde{b}_1 z_1 - (D_t^{q_3-1} e_3(t)) \\ &\quad \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))} - e_3]) + (D_t^{q_3} e_3(t)) \times \frac{(t)^{-(q_3-1)-1}}{\Gamma(-(q_3-1))}]e_3 \\ &\quad + ([D_t^{1-q_4}(D_t^{q_4-1}[\tilde{d}_2 w_2 + \tilde{r}_1 w_1 - (D_t^{q_4-1} e_4(t)) \\ &\quad \times \frac{(t)^{-(q_4-1)-1}}{\Gamma(-(q_4-1))} - e_4]) \\ &\quad + (D_t^{q_4} e_4(t)) \times \frac{(t)^{-(q_4-1)-1}}{\Gamma(-(q_4-1))}]e_4 + \tilde{a}_1 \dot{\tilde{a}}_1 + \tilde{b}_1 \dot{\tilde{b}}_1 + \tilde{c}_1 \dot{\tilde{c}}_1 \\ &\quad + \tilde{r}_1 \dot{\tilde{r}}_1 + \tilde{a}_2 \dot{\tilde{a}}_2 + \tilde{b}_2 \dot{\tilde{b}}_2 + \tilde{c}_2 \dot{\tilde{c}}_2 + \tilde{r}_2 \dot{\tilde{r}}_2 + \tilde{d}_2 \dot{\tilde{d}}_2, \end{aligned} \tag{35}$$

since $\forall q \in [0, 1]$, $(1 - q) > 0$ and $(q - 1) < 0$. Now, using Lemma 1, Eq. (35) reduces to

$$\begin{aligned} \dot{V} &= [\tilde{a}_2(y_2 - x_2) + \tilde{a}_1(y_1 - x_1) - e_1]e_1 \\ &\quad + [\tilde{c}_2 y_2 + \tilde{c}_1 x_1 - e_2]e_2 \\ &\quad + [-\tilde{b}_2 z_2 - \tilde{b}_1 z_1 - e_3]e_3 + [\tilde{r}_2 w_2 + \tilde{r}_1 w_1 - e_4]e_4 \\ &\quad + \tilde{a}_1(-(y_1 - x_1)e_1) + \tilde{b}_1(z_1 e_3) + \tilde{c}_1(-x_1 e_2) \\ &\quad + \tilde{r}_1(-w_1 e_4) + \tilde{a}_2(-(y_2 - x_2)e_1) + \tilde{b}_2(z_2 e_3) \\ &\quad + \tilde{c}_2(-y_2 e_2) + \tilde{r}_2(-x_2 e_3) + \tilde{d}_2(-w_2 e_4), \\ &= -e^T e \le 0. \end{aligned} \tag{36}$$

Since V is positive definite and $\dot{V}$ is negative definite in the neighborhood of the zero solution of the system equation (32), it follows that $\lim_{t \to \infty} \|e(t)\| = 0$. Therefore system (28) can anti–synchronize system (27) asymptotically. This completes the proof. □

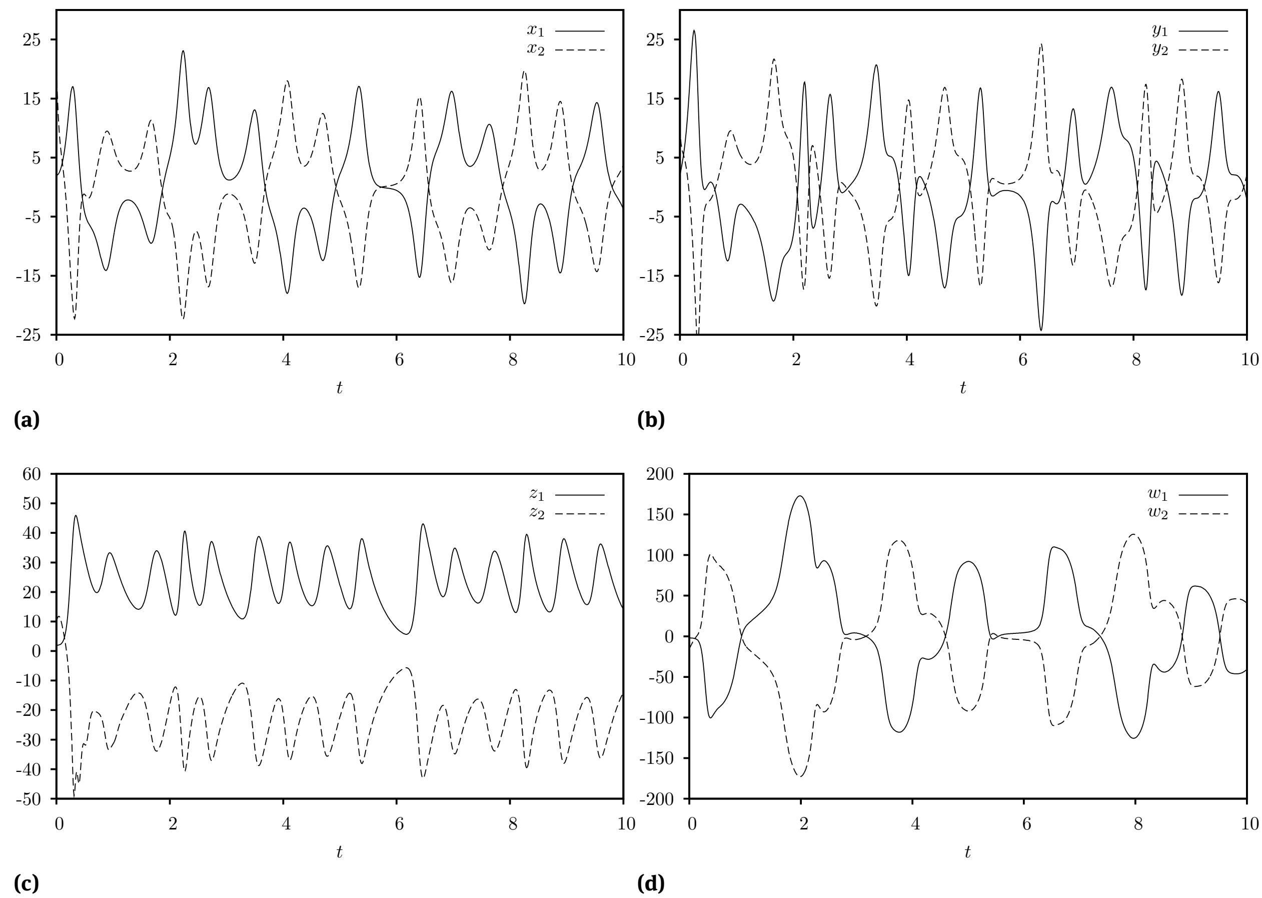


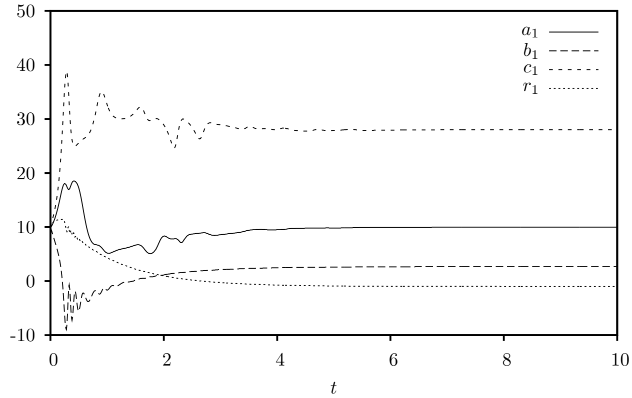
Figure 4: State trajectories of drive system (27) and response system (28): **(a)**: Signals x_1 and x_2; **(b)**: signals y_1 and y_2; **(c)**: signals z_1 and z_2 and **(d)**: signals w_1 and w_2.

6 Numerical simulations

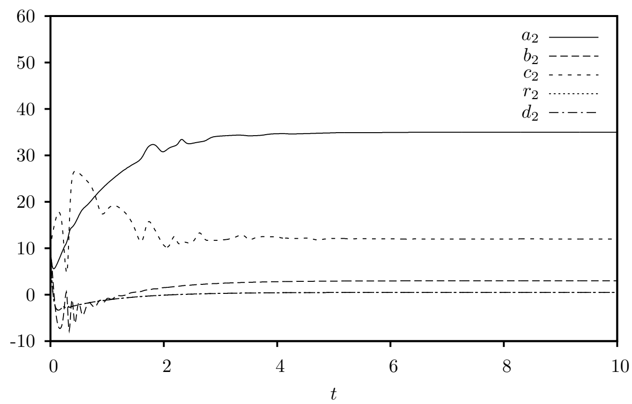
In the numerical simulations, the Adams-Bashforth-Moulton method is used to solve the systems. The fractional order is chosen as $\alpha = 0.97$, and the unknown parameters are chosen as $a_1 = 10, b_1 = 8/3, c_1 = 28, r_1 = -1$ and $a_2 = 35, b_2 = 3, c_2 = 12, r_2 = 0.5, d_2 = 7$, so that both of the systems exhibit a hyperchaotic behavior. The initial values of the fractional-order drive system (27), the fractional-order response system (28) and the estimated parameters are arbitrarily chosen in simulations as $(x_1(0) = 2, y_1(0) = 2, z_1(0) = 2, w_1(0) = -2)$, $(x_2(0) = 20, y_2(0) = 10, z_2(0) = 10, w_2(0) = -15)$, $\hat{a}_1(0) = 10, \hat{b}_1(0) = 10, \hat{c}_1(0) = 10, \hat{r}_1(0) = 10$, and $\hat{a}_2(0) = 10, \hat{b}_2(0) = 10, \hat{c}_2(0) = 10, \hat{r}_2(0) = 10$ respectively. Anti–synchronization of the systems (27) and (28) via the adaptive control law (30) and (31) are shown in Figs. (4)–(5). Fig. (4) displays the state trajectories of the drive system (27) and the response system (28). Fig. (5) (a)–(b) shows that the estimates $\hat{a}_1(t), \hat{b}_1(t), \hat{c}_1(t), \hat{r}_1(t)$ and $\hat{a}_2(t), \hat{b}_2(t), \hat{c}_2(t), \hat{r}_2(t), \hat{d}_2(t)$ of the unknown parameters converge to $a_1 = 10, b_1 = 8/3, c_1 = 28, r_1 = -1$ and $a_2 = 35, b_2 = 3, c_2 = 12, r_2 = 0.5, d_2 = 7$ as $t \to \infty$. Fig. (5) (c) displays the anti-synchronization errors of systems (27) and (28). Fig. (6) shows that the fractional-order hyperchaotic Chen system is controlled to be the fractional-order hyperchaotic Lorenz system.

7 Conclusion

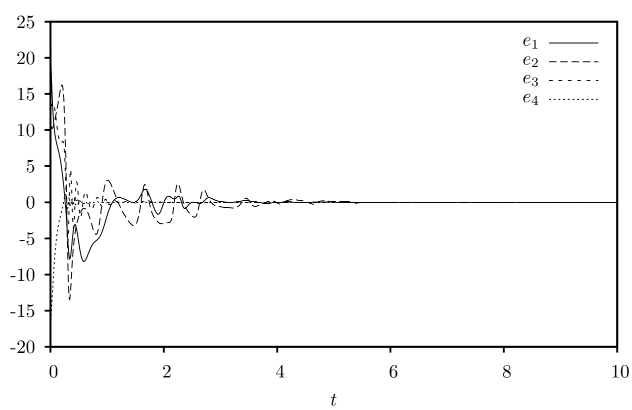
In this paper, we have investigated the anti–synchronization of two different fractional order chaotic and hyperchaotic systems with uncertain parameters. Theoretical analysis was performed to demonstrate the effectiveness of the proposed control strategy. However, we would like to highlight that, in contrast to our method, the active control (cf. [35]) and H_∞ approach (cf. [33, 33]) anti-synchronization are based on exactly known system parameters. As a matter of fact, in real physical systems or



(a)



(b)



(c)

Figure 5: (a)–(b): Changing parameters a_1, b_1, c_1, r_1 and a_2, b_2, c_2, r_2, d_2 of the drive system (27) and the response system (28) with time t. **(c)**: The error signals e_1, e_2, e_3, e_4 of the drive system (27) and the the response system (28) under the controller (30) and the parameters update law (31).

experimental situations some system parameters cannot be exactly known in advance so chaos control and anti-synchronization with uncertain parameters are universal and have received significant attention for their potential applications in prior work. Thus it is much more attractive

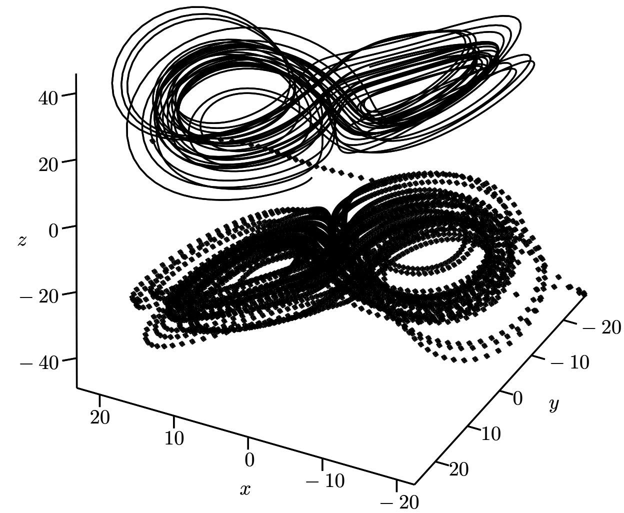


Figure 6: Fractional-order hyperchaotic Lorenz system (solid line) and the controlled fractional-order hyperchaotic Chen system (dotted line) in $x - y - z$ projection.

and challenging to realize the anti–synchronization of two different fractional order chaotic and/or hyperchaotic systems with unknown parameters. We strongly believe that there is high potential in this method and future work is planned to include cost and noise analysis.

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